PROCEEDINGS

OF THE

NATIONAL ACADEMY OF SCIENCES

Volume 1 FEBRUARY 15, 1915 Number 2

CONJUGATE SYSTEMS OF SPACE CURVES WITH EQUAL LAPLACE-DARBOUX INVARIANTS

By E. J. Wilczynski

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO

Read before the Academy, December 7, 1914. Received, December 16, 1914

It is the object of this note to provide a new geometrical interpretation for the condition that the Laplace-Darboux invariants of an equation of the form

$$\frac{\partial^2 z}{\partial x \partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + cz = 0 \tag{1}$$

be equal. These invariants are

$$h = a_x + ab - c, \quad k = b_y + ab - c, \tag{2}$$

where we have put

$$a_x = \frac{\partial a}{\partial x}, \quad b_y = \frac{\partial b}{\partial y}, \tag{3}$$

a notation which we shall employ throughout this paper.

Let $z^{(1)}$, $z^{(2)}$, $z^{(3)}$, $z^{(4)}$ be any four linearly independent solutions of (1), functionally independent in the sense that no two of their three ratios can be expressed as a function of the third. If $z^{(1)}, \ldots, z^{(4)}$ be interpreted as the homogeneous coordinates of a point P of space, variation of the parameters x and y will cause P to generate a non-degenerate surface S which is not a plane. Let us assume further that S is not developable. Then $z^{(1)}, \ldots, z^{(4)}$ will satisfy, besides (1), just one other linear homogeneous differential equation of the second order, of the form

$$z_{yy} = mz_{xx} + nz_x + pz_y + qz, \tag{4}$$

where

$$m = \frac{D''}{D}, \quad D \neq 0, \quad D'' \neq 0, \tag{5}$$

D and D'' being two of the fundamental quantities of S of the second order. The third of these quantities, D', is equal to zero on account of the fact that the curves x = const. and y = const. form a conjugate system on S.

Let us now apply the Laplace transformations to the surface S, by putting

$$z_1 = z_y + az, \quad z_{-1} = z_x + bz. \tag{6}$$

These expressions determine two points P_1 and P_{-1} whose loci give rise to two further surfaces S_1 and S_{-1}. The surface S_1 is the second sheet of the focal surface of the congruence formed by the lines which are tangent to the curves x = const. on S; the surface S_{-1} is connected in similar fashion with the curves y = const. on S.

Consider now the line which joins the points P_1 and P_{-1} of the 1st and -1st Laplace transformed nets. There is one such line for every point P of the original surface S. Consequently, the totality of these lines forms a *congruence*, whose developables we proceed to determine. For this purpose, let us give increments, δx and δy, to x and y. The coordinates of those points of S_1 and S_{-1}, which correspond to the point $z + z_x\, \delta x + z_y \delta y$ of S, will be

$$Z_1 = z_1 + \frac{\partial z_1}{\partial x}\delta x + \frac{\partial z_1}{\partial y}\delta y, \qquad Z_{-1} = z_{-1} + \frac{\partial z_{-1}}{\partial x}\delta x + \frac{\partial z_{-1}}{\partial y}\delta y.$$

Now we find, making use of (1),

$$\begin{aligned} z_1 &= z_y + az, & z_{-1} &= z_x + bz, \\ \frac{\partial z_1}{\partial x} &= -bz_y + (a_x - c)\, z, & \frac{\partial z_{-1}}{\partial x} &= z_{xx} + bz_x + b_x z, \\ \frac{\partial z_1}{\partial y} &= z_{yy} + az_y + a_y z, & \frac{\partial z_{-1}}{\partial y} &= -\, az_x + (b_y - c)\, z. \end{aligned} \tag{7}$$

The homogenous coordinates of an arbitrary point on the line $Z_1\, Z_{-1}$ will be obtained from λZ_1 and μZ_{-1}, and this expression, on account of (7), assumes the form

$$\begin{gathered} \{\lambda\, [a + (a_x - c)\, \delta x + (a_y + q)\, \delta y] + \mu\, [b + b_x \delta x + (b_y - c)\, \delta y]\}\, z \\ + \{\lambda n \delta y + \mu\, (1 + b\delta x - a\delta y)\}\, z_x + \lambda\, \{1 - b\delta x + (a + p)\, \delta y\}\, z_y \\ + (\mu \delta x + \lambda m \delta y)\, z_{xx}. \end{gathered} \tag{8}$$

In order that such a point may also be on the line $P_1\, P_{-1}$, i.e., in order that these two lines may intersect, (8) must reduce to a linear homo-

geneous combination of z_1 and z_{-1}. This will be so, if and only if λ, μ, δx, and δy can be determined subject to the two conditions

$$\lambda m \delta y + \mu \delta x = 0, \tag{9}$$
$$\lambda \left[h + (a_y + q - bn - a^2 - ap)\,\delta y\right] + \mu\left[(b_x - b^2)\,\delta x + k\delta y\right] = 0.$$

If we eliminate the ratio $\lambda : \mu$ from (9), we obtain the differential equation of the developables of the congruence, viz.:

$$h\delta x^2 + \left[a_y + q - bn - a^2 - ap - m\,(b_x - b^2)\right]\delta x\delta y - mk\delta y^2 = 0. \tag{10}$$

This equation may also be regarded as the differential equation of the curves which the developables of the congruence determine on S. The asymptotic curves of S are determined by

$$D\delta x^2 + D''\delta y^2 = 0. \tag{11}$$

Therefore the curves (10) form a conjugate system on S if and only if the simultaneous invariant of (10) and (11) is equal to zero, i.e., if and only if $hD'' - mkD = 0$. But, according to (5), this reduces to $h = k$.

We have, therefore, obtained the following theorem. *Consider a net of conjugate curves on a non-developable surface S. Let P be any point of this net and let P_1 and P_{-1} be the corresponding points of the nets obtained from the given one by the 1st and −1st Laplace transformations. Consider further the congruence of all of the lines such as $P_1\,P_{-1}$. The curves on S, which correspond to the developables of this congruence, will form a conjugate system, if and only if the original net of conjugate curves has equal Laplace-Darboux invariants.*

I wish to add one further remark. Darboux[1] has given a geometric interpretation of the condition $h = k$, different from mine, by means of a certain conic in the plane of P, P_1, P_{-1}.

I have found it advisable to introduce two such conics, which coincide with each other and with that of Darboux in the case $h = k$ and only in that case. By using these conics I have been able, quite recently, to interpret geometrically the condition which Bianchi expresses by saying that a conjugate system is *isothermally conjugate.* These systems have made their appearance in so many problems of differential geometry that such a geometrical interpretation seems to me to be a matter of very great interest. I shall, however, reserve the details of this interpretation for a place, in its appropriate setting, in a paper on the general theory of congruences which I hope to present for publication before long.

[1] *Leçons sur la Théorie générale des Surfaces*, vol. 4, p. 38.